Investigation 2

Linear Models and Equations

In Investigation 1, you used tables, graphs, and equations to study relationships between variables. You found that the strength of a paper bridge depends on both its number of layers and its length. You found that the number of steel pieces needed to build a truss depends on the length of the truss. The number of pieces in a staircase frame depends on the number of steps.

If there is exactly one value of the dependent variable related to each value of the independent variable, mathematicians call the relationship a **function.** For example, the relationship between bridge breaking weight and length is a function. The relationship between the number of steel pieces used and the length of a truss is a function.

Common Core State Standards

8.EE.B.5 Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare two different proportional relationships represented in different ways.

8.EE.C.7b Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.

8.EE.C.8a Understand that solutions to a system of two linear equations in two variables corresponds to points of intersection of their graphs, because points of intersection satisfy both equations simultaneously.

8.F.A.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.

8.F.A.2 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or with verbal descriptions).

Also **8.EE.C.8c, 8.F.A.3, 8.F.B.4, 8.F.B.5, 8.SP.A.1, 8.SP.A.2, 8.SP.A.3, A-SSE.A.1a, A-CED.A.1, A-REI.C.6, F-IF.A.1, F-IF.C.7a, F-BF.A.1a, F-LE.A.2, F-LE.B.5, S-ID.B.6, S-ID.B.6a, S-ID.B.6b, S-ID.B.6c**

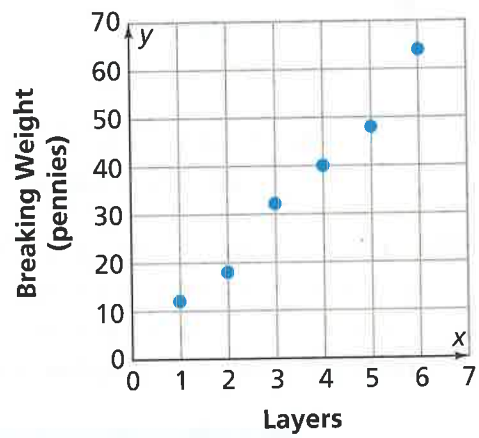
It is often helpful to express functions with equations or formulas. The functions and their equations are called **mathematical models** of the relationships between variables. Equations tell you how to calculate the value of the dependent variable when you know the value of the independent variable. In this Investigation, you will develop skills in writing and using linear equations to model relationships between variables.

2.1 Modeling Linear Data Patterns

Organizing and displaying the data from experiments such as the tests of bridge strength helps you see patterns and make predictions. For linear data, you can usually find a graph and an equation to express the approximate relationship between the variables.

The table and graph below show sample data from Investigation 1.

Bridge Thickness (layers)	Breaking Weight (pennies)
1	12
2	18
3	32
4	40
5	48
6	64



You can see that the points do not lie exactly on a line, but you can draw a line that is a good match for the data pattern. Drawing such a line gives you a model for the data.

- What line would you draw as a model for the data pattern?
- How would you find an equation for this linear function?

Problem 2.1

A The lines in Figure 1 and Figure 2 below represent two different equations or models for the data. The line in Figure 1 connects the points at the left and right ends of the plot. The line in Figure 2 passes among the points but hits none exactly.

1. Which of the two lines seems to fit the data better? Explain your choice.
2. Can you sketch a line that is a better fit than these two?

Figure 1

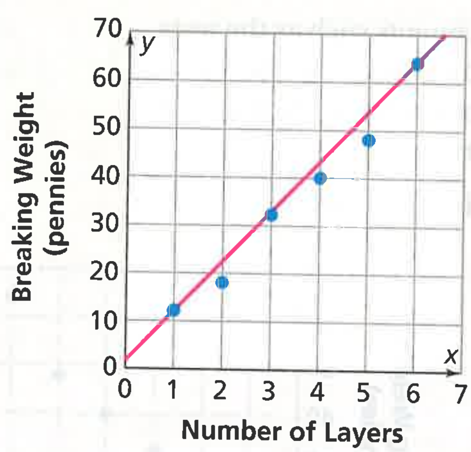
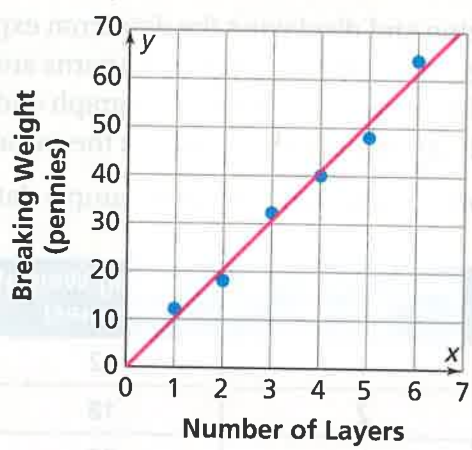


Figure 2



B To find out how accurate each model is, you can calculate errors in the predictions made by the model. Those errors are the differences between the actual data and what each model predicts. Each such error is called a **residual**. Copy and complete the table below for Figure 1 and Figure 2.

Number of Layers	1	2	3	4	5	6
Breaking Weight (pennies)						
Actual	12	18	32	40	48	64
Predicted by Model	▪	▪	▪	▪	▪	▪
Residual (actual − predicted)	▪	▪	▪	▪	▪	▪

1. The first line goes through points (1, 12) and (6, 64). The equation for this line is $y = 10.4x + 1.6$. How would you describe the errors of prediction, or residuals, for this linear model?

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Problem 2.1 continued

2. Sally thinks the equation of the second modeling line is $y = 10x$. Do you agree with Sally? Explain.
3. How would you describe the errors of prediction, or residuals, for Sally's linear model?
4. Do the residuals suggest that one of the models is better than the other? Explain.

C You can find linear models for many situations.

The Student Paint Crew gives weekend and vacation jobs painting houses and apartments to high school and college students. The time a job takes depends on the area to be painted.

Prior jobs give some data relating job area (in units of 1,000 square feet) and time to paint (in hours). The table below shows some of the data.

Area (1,000 sq ft)	1	3	5	8	10
Time (hours)	3	8	12	20	25

1. Plot the given (area, time) data on a graph.
2. Draw a line to match the data pattern.
3. Find the equation of your modeling line.
4. Find the residuals for the model you develop. Explain what they tell you about the accuracy of the linear model.

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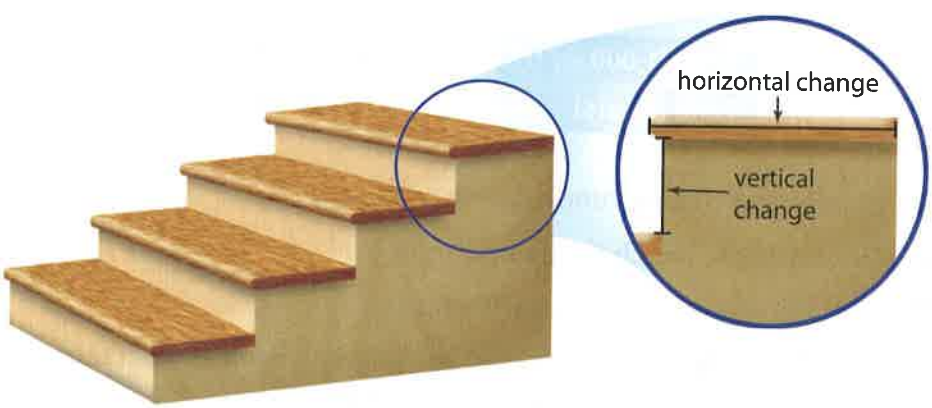
2.2 Up and Down the Staircase
Exploring Slope

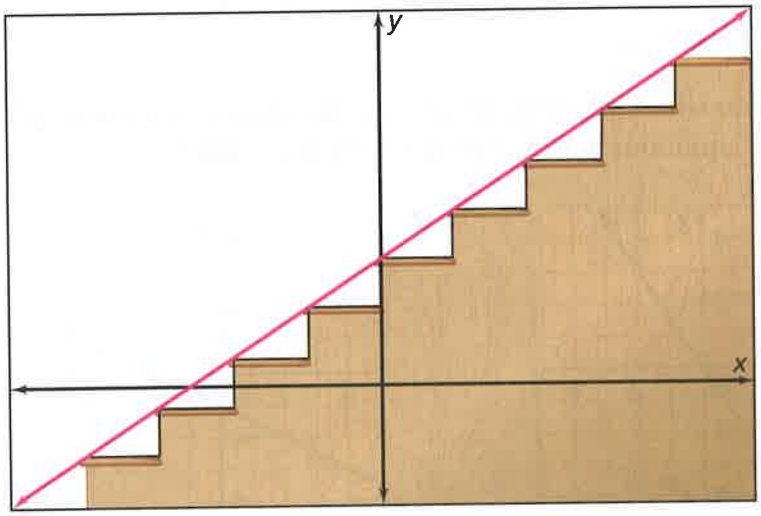
Linear functions are often used as models for patterns in data plots. In *Moving Straight Ahead*, you learned several facts about equations representing linear functions.

- Any linear function can be expressed by an equation in the form $y = mx + b$.
- The value of the coefficient m tells the rate at which the values of y increase (or decrease) as the values of x increase by 1. Since m tells you the change in y for every one-unit change in x, it can also be called the *unit rate.* A unit rate is a rate in which the second number is 1, or 1 of a quantity.
- The value of m also tells the steepness and direction (upward or downward) of the graph of the function.
- The value of b tells the point at which the graph of the function crosses the y-axis. That point has coordinates $(0, b)$ and is called the **y-intercept.**

In any problem that calls for a linear model, the goal is to find the values of m and b for an equation with a graph that fits the data pattern well. To measure the steepness of a linear equation graph, it helps to imagine a staircase that lies underneath the line.



The steepness of a staircase is commonly measured by comparing two numbers, the *rise* and the *run*. The rise is the vertical change from one step to the next, and the run is the horizontal change from one step to the next.



The steepness of the line is the ratio of rise to run. This ratio is the **slope** of the line.

$$\text{slope} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}$$

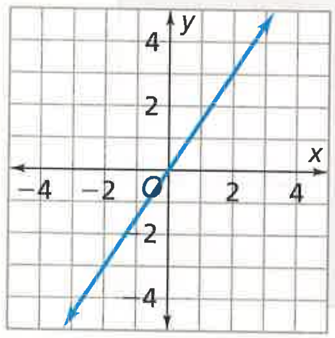
In the diagram of the staircase, the slope of the line is $\frac{2}{3}$. The y-intercept is (0, 5). So the equation for the linear function is $y = \frac{2}{3}x + 5$.

Problem 2.2

Use the data given in each question to find the equation of the linear function relating y and x.

A For the functions with the graphs below, find the slope and y-intercept. Then write the equations for the lines in the form $y = mx + b$.

1.



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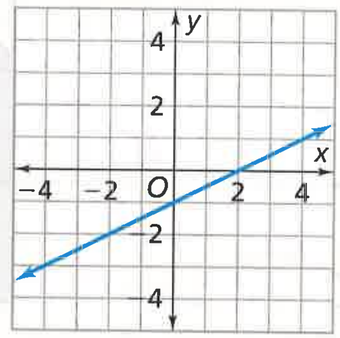

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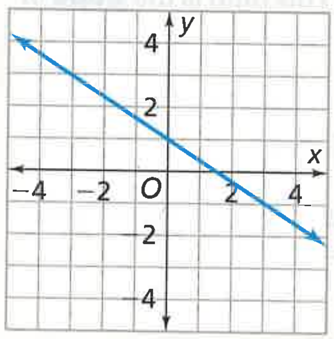

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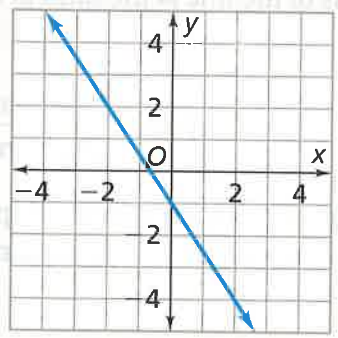

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Problem 2.2 continued

B **1.** Find equations for the linear functions that give these tables. Write them in the form $y = mx + b$.

a.

x	−2	−1	0	1	2
y	−1	1	3	5	7

b.

x	−6	−2	2	6	10
y	−4	−2	0	2	4

2. For each table, find the unit rate of change of y compared to x.

3. Does the line represented by this table have a slope that is greater than or less than the equations you found in part 1(a) and part 1(b)?

x	−1	0	1	2	3
y	4	1	−2	−5	−8

C The points (4, 2) and (−1, 7) lie on a line.

1. What is the slope of the line?

2. Find two more points that lie on this line. Describe your method.

3. Yvonne and Jackie observed that any two points on a line can be used to find the slope. Are they correct? Explain why or why not.

D Kevin said that the line with equation $y = 2x$ passes through the points (0, 0) and (1, 2). He also said the line with equation $y = -3x$ passes through the points (0, 0) and (1, −3). In general, lines with equations of the form $y = mx$ always pass through the points (0, 0) and $(1, m)$. Is he correct? Explain.

E What is the slope of a horizontal line? Of a vertical line?

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2.3 Tree Top Fun
Equations for Linear Functions

Tree Top Fun (TTF, for short) runs adventure sites with zip lines, swings, rope ladders, bridges, and trapezes. The company uses mathematical models to relate the number of customers, prices, costs, income, and profit at its many locations.

Problem 2.3

When finding an equation, it may help to calculate values of the dependent variable for some specific values of the independent variable. Then you can look for a pattern in those calculations. You can use the information given in words, tables of data, and graphs.

A Use what you know about linear equations to work out models for the Tree Top Fun business. Find an equation for each of the linear functions described below.

1. The standard charge per customer at TTF is $25. Write an equation that relates the daily income I to the number n of customers.
2. Each TTF site has operating costs of $500 per day. Write an equation that relates daily profit P to the number n of customers.
3. One TTF site bought a new rope bridge for $4,500. TTF will make monthly payments of $350 until the bill is paid. Write an equation for the unpaid balance B after m monthly payments.

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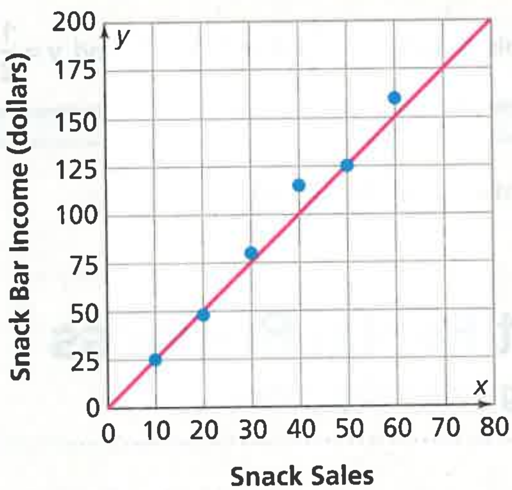
Problem 2.3 continued

B One operator of a Tree Top Fun franchise suggested the group admission fees in the table below.

Number in Group	1	2	3	4	5	10	15	20
Admission (dollars)	75	90	105	120	135	210	285	360

1. Explain how you know the relationship between the admission fee for a group and the number of people in the group is linear.
2. What are the slope and y-intercept of the graph of the data?
3. What equation relates admission fee A to the number n in the group?

C The owners of Tree Top Adventures opened a snack bar at one site. The graph below shows the income from snack sales for six different days. What is the equation of the linear model on the graph?



D Suppose you are asked to write an equation of the form $y = mx + b$ to represent a linear function. What is your strategy for each situation?

1. You are given a description of the function in words.
2. You are given two or more (x, y) values or a table of (x, y) values.
3. You are given a graph showing points with coordinates.

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Problem 2.3 continued

E A state mathematics test asked students to find equations for linear functions. Two students, Dana and Chris, gave the answers below.

1. To find an equation for the line with slope −3 that passes through the point (4, 3), Dana wrote the following steps. Is he correct? Explain.

   $y = -3x + b$, so $3 = -3(4) + b$

   This means $b = 15$ and $y = -3x + 15$.

2. To find an equation for the line that passes through points (4, 5) and (6, 9), Chris wrote the following steps. Is she correct? Explain.

   $m = \frac{6-4}{9-5}$, so $y = \frac{1}{2}x + b$

   This means $5 = \frac{1}{2}(4) + b$, $b = 3$, and $y = \frac{1}{2}x + 3$.

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2.4 Boat Rental Business

Solving Linear Equations

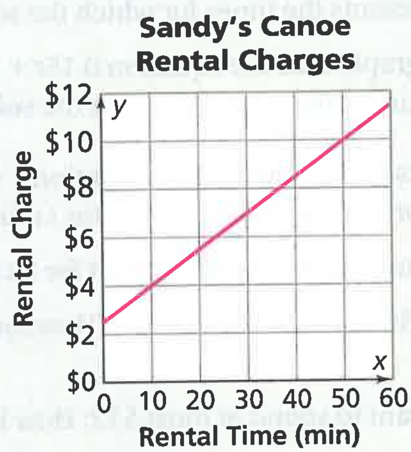
Sandy's Boat House rents canoes at a cost advertised as \$9 per hour for trips on the Red Cedar River. The owner actually gives customers a better deal. She was once a mathematics teacher, and she uses the equation $c = 0.15t + 2.50$ to find the charge c in dollars for renting a canoe for t minutes.

Problem 2.4

When Rashida and Serena applied for jobs at Sandy's, the owner gave them the following test questions to see if they could calculate charges correctly.

A
1. Explain what the numbers in the equation $c = 0.15t + 2.50$ tell you about the situation.
2. How much does it cost to rent a canoe for 25 minutes?
3. A customer is charged \$9.25. How long did he use the canoe?
4. A customer has \$6 to spend. How long can she use a canoe?

B The owner gave Rashida a graph of $c = 0.15t + 2.50$ and asked her how it could be used to estimate answers to Question A. How could Rashida respond?



C The owner asked Serena to explain how she could use the table below to estimate answers to Question A. How could Serena respond?

Canoe Rental Time (min)	10	20	30	40	50	60
Rental Charge (dollars)	4.00	5.50	7.00	8.50	10.00	11.50

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Problem 2.4 continued

D The owner next asked Serena and Rashida to work together to find exact answers, not estimates, for Question A, parts (3) and (4).

1. For part (3) of Question A, the girls solved the linear equation $0.15t + 2.50 = 9.25$. They reasoned as follows:
   - If $0.15t + 2.50 = 9.25$, then $0.15t = 6.75$.
   - If $15t = 6.75$, then $t = 45$.
   - To check the answer, substitute 45 for t: $0.15(45) + 2.50 = 9.25$.

   Are Serena and Rashida correct? How do you know?

2. For Question A, part (4), Rashida said, "The customer can use the canoe for 23.3 minutes if she has \$6." Serena said there are other possibilities—for example, 20 minutes or 15 minutes. Rashida said you can find the answer by solving the **inequality** $0.15t + 2.50 \leq 6$. This inequality represents the times for which the rental costs at most \$6.

   Use the table, graph, and the equation $0.15t + 2.50 = 6$ to find all times for which the inequality is true. Express the solution as an inequality.

E River Fun Boats rents paddle boats. The equation $c = 4 + 0.10t$ gives the charge in dollars c for renting a paddle boat for t minutes.

1. What is the charge to rent a paddle boat for 20 minutes?
2. A customer at River Fun is charged \$9. How long did the customer use a paddle boat?
3. Suppose you want to spend at most \$12. How long could you use a paddle boat?

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2.5 Amusement Park or Movies

Intersecting Linear Models

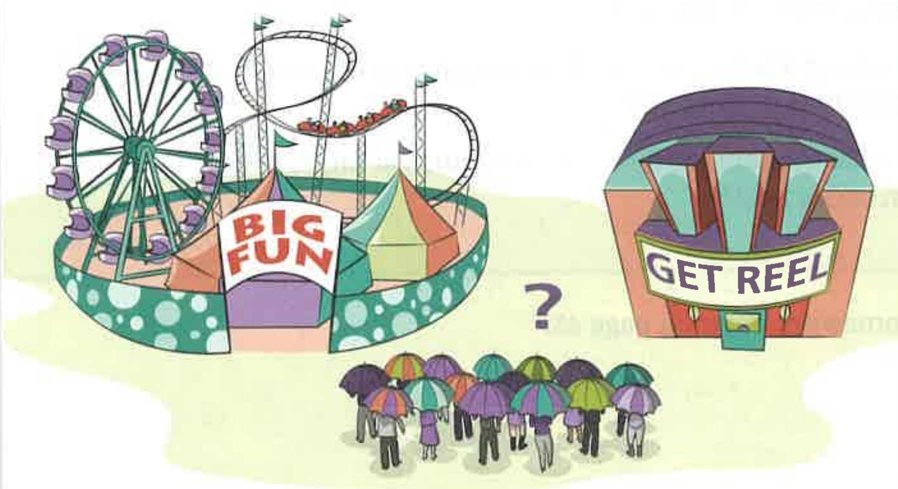
A company owns two attractions in a resort area—the Big Fun amusement park and the Get Reel movie multiplex. At each attraction, the number of visitors on a given day is related to the probability of rain. The company wants to be able to predict Saturday attendance at each attraction in order to assign its workers efficiently.

This table gives attendance and rain-forecast data for several recent Saturdays.

Saturday Resort Attendance

Probability of Rain (%)	0	20	40	60	80	100
Big Fun Attendance	1,000	850	700	550	400	250
Get Reel Attendance	300	340	380	420	460	500

- What equations model the relationships of Big Fun and Get Reel attendance to the probability of rain?
- For what probability of rain will one attraction be more popular than the other?



Problem 2.5

A Use the table to find linear functions relating the probability of rain p to the following quantities.

1. Saturday attendance F at Big Fun
2. Saturday attendance R at Get Reel

Saturday Resort Attendance

Probability of Rain (%)	0	20	40	60	80	100
Big Fun Attendance	1,000	850	700	550	400	250
Get Reel Attendance	300	340	380	420	460	500

B Use your functions from Question A to answer these questions. Show your calculations and explain your reasoning.

1. Suppose there is a 50% probability of rain this Saturday. What is the expected attendance at each attraction?
2. Suppose 475 people visited Big Fun one Saturday. Estimate the probability of rain on that day.
3. What probability of rain gives a predicted Saturday attendance of at least 360 people at Get Reel?
4. Is there a probability of rain for which the predicted attendance is the same at both attractions?
5. For what probability of rain is attendance at Big Fun likely to be greater than at Get Reel?
6. For what probability of rain is attendance at Big Fun likely to be less than at Get Reel?

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